# Fluid-Structure Finite Element Vibrational Analysis

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**A liquid finite element formulation which includes the potential energy due to compression but neglects the density change has been developed. Both kinetic and potential energy are expressed as functions of nodal displacements. Thus, the formulation is similar to that used for structural elements, with the only differences being that 1) the fluid can possess gravitational potential, and 2) the constitutive equations for fluid contain no shear coefficients. Using this approach, structural and fluid elements can be used interchangeably in existing efficient sparse matrix structural computer programs such as SPAR. The theoretical development of the element formulations and the relationships of the local and global coordinates are shown. Solutions of fluid slosh, liquid compressibility, and coupled fluid-shell oscillation problems which were completed using a temporary digital computer program are shown. The frequency correlation of the solutions with classical theory is excellent.**

## Introduction

A FINITE element formulation for calculating the normal modes of a liquid in an arbitrarily shaped flexible container has been obtained. The fluid element was formulated specifically to be compatible with the very efficient methods currently in use for structural vibration analysis, which have been developed by Whetstone.[1,2] The associated computer program (SPAR) has an extensive list of structural elements for modeling large complex structures involving many thousands of degrees of freedom. Incorporation of liquid elements in the program is planned. The objective is to demonstrate the theoretical adequacy of a method for studying the vibrational characteristics of structural systems containing a large amount of liquid. The system, such as the liquid propellant space vehicle, can be modeled as a network of structural and liquid elements.

To achieve this goal, velocity coordinates and compressible formulation were used. Therefore, the only differences between a structural element and a fluid element are the constitutive equations and gravitational potential energy. The resulting matrices for the equations are sparse, and all nodes in the fluid have an identicial number of degrees of freedom to provide simple topology. The step of elimination of interior nodes in the solution procedures of previous studies such as those of Luk,[3] Khabbaz,[4] and Guyan[5] is avoided. Consequently, the formulated eigenvalue problem becomes the solution of a larger, sparse matrix equation instead of a smaller, full matrix equation resulting from the elimination of internal nodes.

Numerical accuracy is achieved by using double precision for some calculations. Several simple finite elements were formulated to evalute the concept. Similarly, a simple solution technique was chosen for demonstration. The examples were chosen to verify the accuracy of the method rather than the generality of the configuration which can be analyzed.

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## Formulation of the Eigenvalue Problem

Nodal displacements are used as the generalized coordinates of the formulated eigensystem. In general, the motion of a liquid node is represented by three displacement components, whereas the motion of a structural node consists of three displacement components and three rotational displacements. Kinetic and potential energies of an element are defined first in terms of nodal displacements in the element coordinate system. A transformation then is performed to express these energies in terms of nodal displacements in the local joint coordinate systems. The assembly of all element matrices using the direct stiffness method will give the total kinetic and potential energies of the system in the matrix form as

$$T = \tfrac{1}{2}\dot{Q}^T M \dot{Q} \tag{1}$$

$$V = \tfrac{1}{2}Q^T K Q \tag{2}$$

where $M$ and $K$ are the mass and stiffness matrices of the system, respectively. The generalized coordinate vector $Q$ consists of subvectors $q^j$ $(j=1,2,\ldots,N)$, where $N$ is the total number of unconstrained or partially constrained nodes of the coupled system. The subvector $q^j$ can be expressed as

$$\begin{bmatrix} (q_x^j)_e \\ (q_y^j)_e \\ (q_z^j)_e \end{bmatrix}, \quad \begin{bmatrix} (q_x^j)_s \\ (q_y^j)_s \\ (q_z^j)_s \\ (\omega_x^j)_s \\ (\omega_y^j)_s \\ (\omega_z^j)_s \end{bmatrix}, \text{ or } \quad \begin{bmatrix} (q_y^j)_e \\ (q_z^j)_e \\ (q_x^j)_s \\ (q_y^j)_s \\ (q_z^j)_s \\ (\omega_x^j)_s \\ (\omega_y^j)_s \\ (\omega_z^j)_s \end{bmatrix}$$

for a liquid, a shell, or a container node, respectively. A container node is defined as a node where a liquid node and a structural node are coincident geometrically. The notations $[(q_x^j)_e, (q_y^j)_e, (q_z^j)_e]$ or $[(q_x^j)_s, (q_y^j)_s, (q_z^j)_s]$ represent the displacement components of the $j$th liquid or the $j$th structural node, respectively,

and $[(\omega_x^j)_s, (\omega_y^j)_s, (\omega_z^j)_s]$ are the rotational components of the $j$th structural node.

Fluid elements considered in this study are those to be used to represent a liquid, such as water, liquid oxygen, or hydrogen, in a flexible container. The liquids are compressible, but their mass density variation is negligible. The potential energy of the element is the summation of two energies: 1) gravitational potential, and 2) dilational potential. In the absence of shearing effects, the kinetic, gravitational, and dilational energies of a liquid finite element can be calculated from the following integrals:

$$T_\ell = \tfrac{1}{2}\rho \int (\dot{u}^2 + \dot{v}^2 + \dot{w}^2)\,dV \tag{3}$$

$$V_\ell = \tfrac{1}{2}\rho \int g\delta^2 dS \tag{4}$$

and

$$S_\ell = \tfrac{1}{2}\kappa \int \theta^2 dV \tag{5}$$

respectively. The notations used in Eqs. (3-5) are defined as follows:

| | |
|---|---|
| $u,v,w$ | = displacement components of a fluid point in the $x, y, z$, directions, respectively |
| $g$ | = equivalent gravitational acceleration component normal to the surface |
| $\delta$ | = displacement normal to the surface |
| $\rho$ | = mass density |
| $\kappa$ | = bulk modulus |
| $\theta$ | = volume change of an element |

## Element Formulation

An element is formulated by selecting a displacement function which is linear along all boundary edges of the element. Kinetic and potential energies of the element are expressed in integral forms which are evaluated in an intrinsic coordinate system or its natural coordinate system. A subsequent coordinate transformation is performed such that the energies can be written in terms of nodal displacements in the local joint coordinate system. In this way, a large and complex fluid-structural model represented by various kinds of finite elements can be assembled readily to satisfy the boundary conditions and interface constraint requirements while forming the system mass and stiffness matrices.

Formulation of structural elements can be found in many publications: Zienkiewicz[6] or Norrie,[7] for example. A quadrilateral plate element routine used in this study was developed by Yen.[8] The following discussion on element formulation will be limited to a fluid tetrahedron and a fluid hexahedron. Other fluid elements, such as pentahedron and two-dimensional triangular and quadrilateral elements, are found to be the special cases of these elements.

### Fluid Tetrahedron

The displacement fluid of a fluid tetrahedron is assumed to be linear throughout the element (Fig. 1). For instance, the displacement component in the $x$-direction of a fluid point can be expressed as

$$u = \boldsymbol{x}\boldsymbol{a} \tag{6}$$

where

$$\boldsymbol{x} = [1\ x\ y\ z]$$

and $\boldsymbol{a}$ is a $4\times 1$ coefficient matrix to be determined. Substituting the nodal displacements and nodal position coordinates into Eq. (6),

$$\alpha_x = \boldsymbol{X}\boldsymbol{a} \tag{7}$$

where

$$\alpha_x = \begin{bmatrix} \alpha_x^1 \\ \alpha_x^2 \\ \alpha_x^3 \\ \alpha_x^4 \end{bmatrix} \quad \text{and} \quad X = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & x_2 & 0 & 0 \\ 1 & x_3 & y_3 & 0 \\ 1 & x_4 & y_4 & z_4 \end{bmatrix}$$

The notation $\alpha_x^j$ represents the $x$-direction component of the $j$th node displacement. Inverting the coefficient matrix $X$ and substituting back to Eq. (6), the displacement $u$ can be written in terms of nodal displacements $\alpha_x^j$

$$u = \boldsymbol{x}\boldsymbol{X}^{-1}\alpha_x \tag{8}$$

Similarly,

$$v = \boldsymbol{x}\boldsymbol{X}^{-1}\alpha_y, \quad w = \boldsymbol{x}\boldsymbol{X}^{-1}\alpha_z \tag{9}$$

Substituting Eqs. (8) and (9) into Eq. (3), the kinetic energy of a fluid tetrahedron may be expressed as

$$T_\ell = \tfrac{1}{2}\rho[\dot{\alpha}_x{}^T \boldsymbol{m}\dot{\alpha}_x + \dot{\alpha}_y{}^T \boldsymbol{m}\dot{\alpha}_y + \dot{\alpha}_z{}^T \boldsymbol{m}\dot{\alpha}_z] \tag{10}$$

where

$$\boldsymbol{m} = (\boldsymbol{X}^{-1})^T \left(\int \boldsymbol{x}^T\boldsymbol{x}\,dV\right)\boldsymbol{X}^{-1}$$

If the nodal displacement vectors of Eq. (10) are rearranged, the kinetic energy can be written in the following form:

$$T_\ell = \tfrac{1}{2}\,\rho\dot{\boldsymbol{\alpha}}^T\boldsymbol{M}_\ell\dot{\boldsymbol{\alpha}} \tag{11}$$

where $\boldsymbol{M}_\ell$ is the element mass matrix which consists of the elements of the matrix $\boldsymbol{m}$, and

$$\boldsymbol{\alpha}^T = [\alpha_x^1, \alpha_y^1, \alpha_z^1, \alpha_x^2, \alpha_y^2 \ldots \alpha_z^4]$$

The gravitational energy of a fluid tetrahedron is calculated by expressing the surface displacement in terms of nodal displacements in an intrinsic Cartesian coordinate system. Figure 2 shows a boundary face of the element opposite to node 1. The new position coordinate of notes 2, 3, and 4 are

$$\alpha_3 = \ell_4,\ \alpha_4 = \ell_5\cos\theta,\ \beta_4 = \ell_5\sin\theta$$

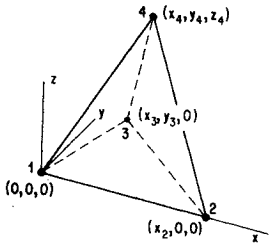


**Fig. 1 Fluid tetrahedron with coordinates.**

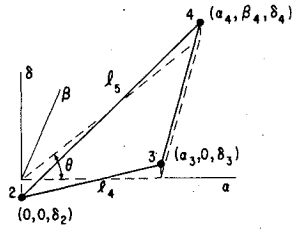


**Fig. 2 A boundary face of the fluid tetrahedron.**

and

$$\delta_j = [n_\alpha \; n_\beta \; n_\delta] \begin{bmatrix} \alpha_x^j \\ \alpha_y^j \\ \alpha_z^j \end{bmatrix} \qquad (j=2,3,4)$$

where $\ell_4$ and $\ell_5$ are distances between two nodes, and $\theta$ is the angle between boundary edges. Because of the linear nature of the displacement field, the surface displacement can be expressed in terms of nodal displacements:

$$\delta = f\alpha \tag{12}$$

where $f$ is a $1 \times 12$ coefficient matrix which is determined by the geometry of the element.

Substituting Eq. (12) into Eq. (4) and performing the surface integration, the gravitational energy of a fluid tetrahedron can be obtained by summing up the contributions of all boundary faces

$$V_\ell = \tfrac{1}{2}\rho \; \alpha^T G \alpha \tag{13}$$

where $G$ is the element gravitation matrix which is determined by the gravitational acceleration and element geometry.

The dilational energy of a fluid tetrahedron can easily be found by direct differentiation:

$$\theta = a_2 + a_3 + a_4 = k\alpha \tag{14}$$

Substituting Eq. (14) into Eq. (5), one finds

$$S_\ell = \tfrac{1}{2} \; \rho \alpha^T K_\ell \alpha \tag{15}$$

where

$$K_\ell \equiv \kappa V_e k^T k$$

is the element dilation matrix, and $V_e$ denotes the volume of the element.

**Fluid Hexahedron**

The following displacement field is used to define the characteristics of a fluid hexahedron in its natural coordinate system (Fig. 3):

$$u = \begin{bmatrix} \xi & 0 & 0 \\ 0 & \xi & 0 \\ 0 & 0 & \xi \end{bmatrix} \begin{bmatrix} \alpha_x \\ \alpha_y \\ \alpha_z \end{bmatrix} \tag{16}$$

where

$$\xi = [\xi_1, \; \xi_2, \; \xi_3, \; \xi_4, \; _5, \; \xi_6, \; \xi_7, \; \xi_8]$$

and where

$$\xi_1 = (1-\xi) \; (1-\eta)(1-\zeta)/8 \tag{17a}$$

$$\xi_2 = (1+\xi)(1-\eta)(1-\zeta)/8 \tag{17b}$$

$$\xi_3 = (1+\xi)(1+\eta)(1-\zeta)/8 \tag{17c}$$

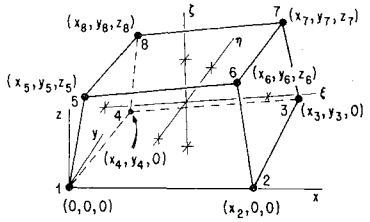


**Fig. 3 Fluid hexahedron with coordinates.**

$$\xi_4 = (1-\xi)(1+\eta)(1-\zeta)/8 \tag{17d}$$

$$\xi_5 = (1-\xi)(1-\eta)(1+\zeta)/8 \tag{17e}$$

$$\xi_6 = (1+\xi)(1-\eta)(1+\zeta)/8 \tag{17f}$$

$$\xi_7 = (1+\xi)(1+\eta)(1+\zeta)/8 \tag{17g}$$

$$\xi_8 = (1-\xi)(1+\eta)(1+\zeta)/8 \tag{17h}$$

The natural coordinates $(\xi, \eta, \zeta)$ are related to the element coordinates $(x,y,z)$ by the following equation[6]:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & x_8 \\ 0 & y_3 & y_4 & y_5 & y_6 & y_7 & y_8 \\ 0 & 0 & 0 & z_5 & z_6 & z_7 & z_8 \end{bmatrix} \begin{bmatrix} \xi_2 \\ \xi_3 \\ \xi_4 \\ \xi_5 \\ \xi_6 \\ \xi_7 \\ \xi_8 \end{bmatrix} \tag{18}$$

The displacement function of Eq. (16) varies linearly along the boundary edges of the hexahedral element. Substituting this equation in Eqs. (3) and (4) and performing the volume and surface integrals, respectively, by the gaussian quadrature formula, the kinetic and gravitational energies of the element can be written in the same form as those of Eqs. (11) and (13).

The dilational matrix of a hexahedral element is obtained by minimizing the complementary energy of the element. It can be found that the derivation for a liquid element by this method is a special case of the method of Pian.[9] The stresses of the element are

$$\sigma_{xy} = \sigma_{yz} = \sigma_{zx} = 0$$

and

$$\sigma_{xx} = \sigma_{yy} = \sigma_{zz} = \kappa\theta = \beta_1 = \text{const}$$

and the dilational energy of the element becomes

$$S_\ell = \tfrac{1}{2}(V_e/\kappa) \; \beta_1^2 \tag{19}$$

Consequently, the complementary energy of the element can be written as

$$S_c = \tfrac{1}{2}(V_e/\kappa) \; \beta_1^2 - \beta_1 R\alpha \tag{20}$$

where $R$ is a known $1 \times 24$ coefficient matrix that is determined from the assumed displacement field and the given element geometry. By applying the principle of minimum complementary energy, the constant $\beta_1$ can be expressed in terms of the nodal displacements:

$$\beta_1 = (\kappa/V_e) R\alpha \tag{21}$$

Substituting Eq. (21) into Eq. (19), the dilational energy of a fluid hexahedron can be rewritten in a similar form of Eq. (15) with

$$K_\ell = (\kappa/V_e) R^T R$$

## Solutions

For these investigations, a special computer program was developed for the solution of the matrix equations. This

program is adequate to evaluate the feasibility of the selected approach and to analyze single tank configurations. The program must be regarded as temporary, however, since greater speed, greater size capacity, and generality ultimately are desired. Under a separate effort, liquid elements are planned to be incorporated into the SPAR program which solves extremely large sparse matrix systems. The advantages of the sparse matrix technique can be demonstrated by taking a $10\times10\times10$ node cube as an example. Thus the matrix would have 1000 nodes, with 488 on the surfaces. The interconnect level (or bandwidth in this case) is 111. The number of arithmetic operations involved in processing a full matrix is proportional to $n^3/3$, where $n$ is the matrix size, whereas the number of operations for a sparse matrix is $nb^3/2$, where $b$ is the interconnect level. For the example chosen, $488^3/3=38.7\times10^6$, whereas $1000\times111^2/2=6.16\times10^6$. Therefore, a sparse matrix method should be able to solve the problem about 6 times as fast as a method which reduced out the internal degrees of freedom. Detailed discussions on the operations for full and sparse matrices can be found in Ref. 10 and 11, respectively.

The solutions to the matrix equations will give eigenvalue frequencies over a large range of frequencies. The lowest possible frequency is zero, corresponding to a condition with no potential energy. Multiple modes usually will occur, representing all patterns of circulation possible with the model degrees of freedom.

The simplest example of a circulation mode in two dimensions is shown in Fig. 4. Here the upper surface is free, but no displacement occurs. For this simple case, the mode shape can be hand-checked, assuming incompressibility. The number of circulation modes increases rapidly with model size and has a significant effect on the choice of a solution technique. Certainly, methods which calculate modes individually in increasing order of frequency are not practical.

Some care must be exercised in the selection of element configurations. Inadvertent constraints are possible. For example, in Fig. 5, the point $A$ cannot move perpendicular to line $AB$ without changing the volume of either element $\alpha$ or $\beta$. No constrained cases have been discovered where two elements meet at the corner, such as point $C$. However, no rigorous criteria have been developed to assure unconstrained configurations.

The next modes to occur in the frequency spectrum are the slosh modes. For elements of the typical sizes used for these investigations with lengths of 0.1 to 1.0 m, the coefficients of the gravitational potential energy matrix $V_f$ are six orders of magnitude less than the coefficients of the dilational potential energy matrix $S_f$. Round-off problems were avoided by the use of double precision arithmetic. The number of slosh modes always will be one less than the number of nodes on the free surface. The highest modes of the system will be those representing elastic deformations of the liquid and/or container. A limited number of configurations have been analyzed for elastic modes.

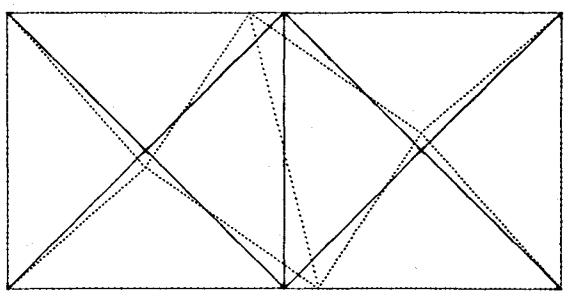

**Fig. 4 Circulation mode of a simple two-dimensional configuration.**

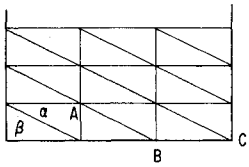


**Fig. 5 Constrained element configuration.**

## Results

Although the purpose of the analysis is to calculate the modes of a truly arbitrary configuration, simple configurations with known solutions are required for accuracy checks. A typical result for sloshing in a right circular cylinder is shown in Fig. 6. Water is the fluid, the radius is 1.0 m, and the depth is 2.0 m.

A quarter model is used to reduce the matrix size, the symmetric constraints applied to the near plane in the figure and antisymmetric constraints on the other symmetry plane. The frequency of 0.6664 Hz, calculated by the program, is 1.5% lower than the classical solution value of 0.6762 Hz for the lowest antisymmetric slosh mode of the cylinder. This mode is actually the 49th mode of the model used. This model has 77

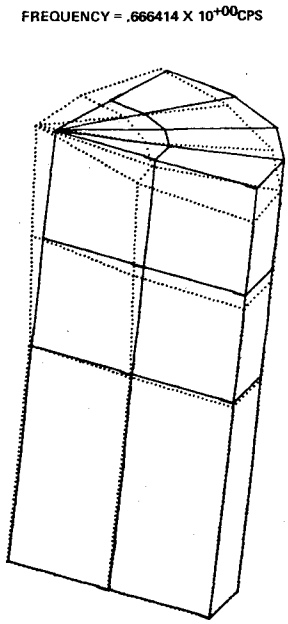


**Fig. 6 First antisymemtric slosh mode in a cylindrical tank.**

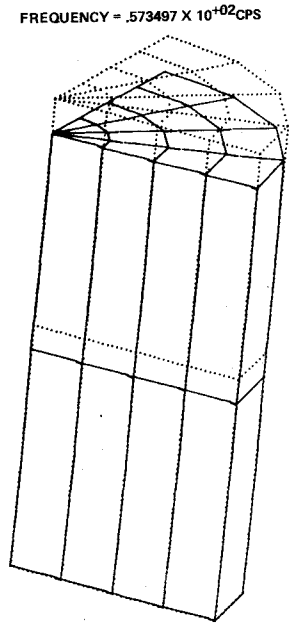


**Fig. 7 Liquid compressibility mode in a rigid cylindrical tank.**

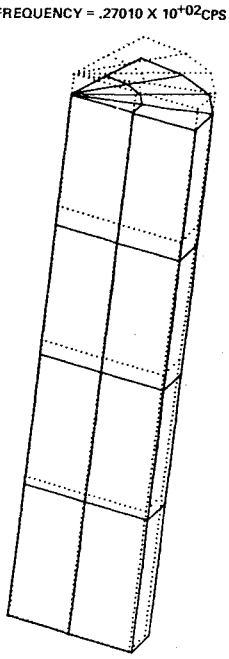


**Fig. 8 Axial mode of a cylindrical tank with flexible wall.**

unrestrained degrees of freedom, and the modal solution required 1 min and 21 sec of UNIVAC 1108 CPU time.

Some care must be taken to avoid round-off error. A parametric investigation was made, reducing the gravitational acceleration ($g$) for this case. Reduction of $g$ by a factor of 10 produced little change in accuracy. However, a reduction by a factor of 100 caused large errors as the gravitational potential became lost in the round-off of the elastic potential energy. The same effect could occur if element dimensions were reduced.

The accuracy of the fluid compressibility model was checked using the same rigid cylindrical tank as in the previous example. The resultant mode is shown in Fig. 7. The frequency of 57.3 Hz is 2.6% higher than the frequency of 55.9 calculated using classical methods.

The case for liquid in a flexible-walled tank was checked using a test case of water in a cylinder 1.0 m in radius and 4.0 m deep. The wall has a thickness of 0.002 m, Young's modulus of $0.20 \times 10^{12}$ N/m$^2$ and a mass density of 7920 kg/m$^3$. The bottom is rigid and not connected to the side wall. The element configuration and the nodal displacements are shown in Fig. 8. The calculated frequency by the finite element method is 27.01 Hz compared to a frequency of 26.69 Hz calculated by classical methods, given an error of only 1.2%.

## Conclusions

The development of the quasicompressible tetrahedral and hexahedral elements has been successful. Results to date indicate excellent accuracy for the method. The temporary computer program is adequate for the analysis of slosh or hydroelastic modes of a single tank. The elements need to be incorporated into a computer program which uses sparse matrix techniques for the solution of larger, more complex problems in reasonable computer times.

Structural systems containing a large amount of liquids can be modeled as networks of structural and liquid elements. The vibrational characteristics of these coupled systems can be studies through finite-element computer programs. An alternative to the conventional approach of representing the sloshing liquid by an equivalent mass-spring system,[12] which sometimes involves questionable assumptions, is presented. Also, no assumption of symmetry is made in the theory.

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